DIFFERENTLY KNOTTED SYMPLECTIC SURFACES IN D^4 BOUNDED BY THE SAME TRANSVERSE KNOT

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1. INTRODUCTION

This paper is concerned with symplectic surfaces in the four-dimensional ball. More precisely, we consider embedded connected surfaces $S \subset D^4$ whose boundary $\partial S = S \cap \partial D^4 \subset S^3$ is a transverse knot. We prove:

Theorem. *There are two symplectic surfaces S_1 and S_2 which bound the same transverse knot, have the same topology (as abstract surfaces), and such that $\pi_1(D^4 \setminus S_1)$ is not isomorphic to $\pi_1(D^4 \setminus S_2)$.*

This builds on a previous example ([2], §5) of surfaces bounding the same transverse link, which had different topology (one was connected, and the other was not). We add the same piece to both these examples to construct ours.

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2. BACKGROUND

It is well-known [4] that the closure of any braid $\beta \in Br_m$ is naturally a transverse link. This has been used to construct examples of transverse links which lie in the same topological isotopy class, but are not isotopic as transverse links [5]. A factorization of β is an expression
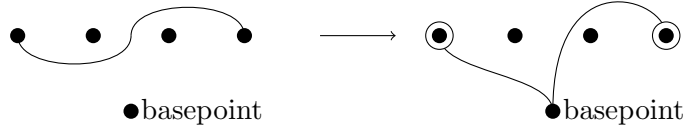
$$\beta = \sigma_1 \cdots \sigma_k$$

where each σ_i is conjugate to one of the standard Artin generators of Br_m. Every factorization of β describes a symplectic surface $S \subset D^4$ whose boundary is the transverse link associated to β (see e.g. [3]). S is connected if and only if the images of $\sigma_1, \ldots, \sigma_k$ in the symmetric group Sym_m act transitively. The Euler characteristic of S is $m - k$; hence the topological type of the abstract surface S depends only on β, and not on the factorization.

There is a well-known method [8] for computing a presentation $\pi_1(D^4 \setminus S)$, as a quotient of the fundamental group of the m-punctured disc, which is $F_m = \langle x_1, \ldots, x_m \rangle$. Every word σ_i in the factorization yields a relation. This is best represented graphically as follows. Conjugates of the standard Artin generators correspond bijectively to embedded paths in the disc (up to isotopy) joining two of the punctures.

Given any such path, one fattens it into a figure-eight loop enclosing the punctures at its endpoints. From a path γ joining two punctures p_i and p_j, one obtains a loop $\gamma' c_i (\gamma')^{-1} c_j^{-1}$, where c_i and c_j are counterclockwise circles around the punctures and γ' is an appropriate segment of γ.

This identifies a conjugacy class in F_m, any element of which can be taken as the relation implied by this path. For consistency or ease of computation we may choose any convenient basepoint to determine what this relation is.

For example, let a, b, and c denote the Artin generators in Br_4. The word $(ac)b(ac)^{-1}$ yields the following:

$$x_1 = x_3^{-1} x_4 x_3$$

3. The Examples

Let a, b, and c be the Artin generators in Br_4. Appending the factor $a^3 b a^{-3}$ to the examples given in [2] §5 yields the following factorizations:

$$(1) \qquad (c^{-2}b)a(c^{-2}b)^{-1} \cdot b \cdot (ac^3)b(ac^3)^{-1} \cdot$$
$$(ac^5 b^{-1})a(ac^5 b^{-1})^{-1} \cdot c \cdot c \cdot a^3 b a^{-3}$$

$$(2) \qquad b \cdot (ac)b(ac)^{-1} \cdot (ac)b(ac)^{-1} \cdot (ac)^2 b(ac)^{-2} \cdot$$
$$(ac)^2 b(ac)^{-2} \cdot (ac)^3 b(ac)^{-3} \cdot a^3 b a^{-3}$$

It was checked in [2] that these factorizations are of the same braid. Their images in Sym_4 are:

$$(3) \qquad (13) \cdot (23) \cdot (14) \cdot (24) \cdot (34) \cdot (34) \cdot (13)$$
$$(4) \qquad (23) \cdot (14) \cdot (14) \cdot (23) \cdot (23) \cdot (14) \cdot (13)$$

The transpositions (13), (14), and (23) are present in both and suffice to generate Sym_4, so S_1 and S_2 are both connected. The product is the cyclic permutation (1234), so the boundaries ∂S_1 and ∂S_2 are also connected.

From factorization (2), we have the following relations:

$$b \qquad\qquad x_2 = x_3$$
$$c \qquad\qquad x_3 = x_4$$
$$(c^{-2}b)a(c^{-2}b)^{-1} \qquad\qquad x_1 = x_2^{-1} x_4 x_3 x_4^{-1} x_2 = x_3$$

Given these relations, the relations arising from the rest of the factors simplify to identity. With, $x_1 = x_2 = x_3 = x_4$, $\pi_1(D^4 \setminus S_1)$ must be $\mathbb{Z}$.

From factorization (3), we have:

$$b \qquad\qquad x_2 = x_3$$
$$(ac)b(ac)^{-1} \qquad\qquad x_1 = x_3^{-1}x_4x_3$$
$$a^3ba^{-3} \qquad\qquad x_3 = x_1^{-1}x_2^{-1}x_1x_2x_1$$

This last relation can be rewritten, using $x_2 = x_3$, as:

$$x_1x_2x_1 = x_2x_1x_2$$

As with the previous example, the remaining factors yield relations that simplify to identity given these three. Thus $\pi_1(D^4 \setminus S_2) = Br_3$.

4. A NOTE ON DOUBLE BRANCHED COVERS OF D^4

Given any connected surface $S \subset D^4$, one can form the double cover of D^4 branched along S, which we denote by M. If S is symplectic and its boundary is a transverse link, M is an exact symplectic manifold with contact type boundary [3].

The fundamental group $\pi_1(D^4 \setminus S)$ comes with a canonical homomorphism $\phi : \pi_1(D^4 \setminus S) \to \mathbb{Z}$. The meridian element x, determined up to conjugacy, satisfies $\phi(x) = 1$. To compute $\pi_1(M)$, one takes the subgroup $\pi_1(D^4 \setminus S)' = \phi^{-1}(2\mathbb{Z})$, and then divides out by x^2 (this is a simple application of the Seifert-van Kampen theorem).

Take the examples S_1, S_2 from the previous Section, and let M_1, M_2 be the associated double branched covers. It is easy to see that $\pi_1(M_1)$ is trivial. In the second case, we have a homomorphism $\pi_1(D^4 \setminus S_2) \cong Br_3 \to Sym_3$, which restricts to $\pi_1(D^4 \setminus S_2)' \to A_3 \cong \mathbb{Z}/3$. Since x^2 goes to zero under this, we get an induced homomorphism $\pi_1(M_2) \to \mathbb{Z}/3$. One easily checks that this is surjective. Hence, M_1 and M_2 are two different exact symplectic fillings of the same contact three-manifold $\partial M_1 = \partial M_2$.

For other such examples and further discussion, see [1], [6], and [7].

5. CLOSING REMARKS

Numerical evidence suggests that, for odd $s \geq 3$, using a^sba^{-s} in place of a^3ba^{-3} makes $\pi_1(D^4 \setminus S_2) = \langle x, y \mid x^2 = y^s \rangle$ while $\pi_1(D^4 \setminus S_1)$ remains as $\mathbb{Z}$. These groups are distiguished from each other by the number of homomorphisms from them to the dihedral groups.

Electronic resources (Python code) for reproducing the numerical results can be found online at http://www-math.mit.edu/~seidel/geng/.

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